Robot Swarms for Exploration and Triangulation of Unknown Environments

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Abstract

We consider a robot swarm in an unknown polygon. All robots have only a limited communication range. We then look for a triangulation of the polygon using the robots as vertices such that the number of robots used for the triangulation is minimized. All edges in the triangulation have a length smaller than the communication range. For this *Online Minimum Relay Triangulation Problem*, we present a lower bound of $\frac{9}{8}$ on the competitive ratio for any online algorithm. Moreover, we give an algorithm that is $\frac{21}{4}$-competitive for simple polygons and 6-competitive for general polygons.

1 Introduction

Exploring and guarding polygonal regions are classical problems that have been investigated for decades. Hoffmann et al. [6] considered the online exploration of simple polygons with unlimited vision, Icking et al. [7] and Fekete et al. [5] exploration with limited and time-discrete vision, respectively. Exploration with both limited and time-discrete vision is presented by Fekete et al. [4]. Placing stationary guards was first considered by Chvátal [1], see also O'Rourke [8].

In this paper, we combine both problems, motivated as follows. Consider a static sensor network that needs to react to different scenarios by adding further mobile sensors, e.g. sensor nodes attached to mobile robots as in Figure 1. Typically, these sensor nodes have a limited communication range, and no common orientation or coordinate system is available; furthermore, the expanded network has to be well connected, asking for a triangulated network.

Classical triangulation problems (see, e.g., [8, 2]) ask for a triangulation of all vertices of a polygon, but allow arbitrary length of the edges in the triangulation. This differs from our problem, in which a limitation on the edge length (given by the communication length) is given. Placing vertices of the triangulation

Figure 1: A robot swarm consisting of iRobot Roombas.

(robots) on arbitrary positions in the polygon in order to achieve this limited length is closely related to the *relay placement problem*, in which a set of sensors is to be connected by relays with limited range [3]. We use the terms *robots* and *relays* synonymously.

The rest of the paper is organized as follows. The following Section 2 provides definitions. We present a lower bound on the competitive ratio in Section 3. In Section 4 we describe a 6-competitive algorithm for polygons (with holes) and prove that this algorithm is $\frac{21}{4}$-competitive for simple polygons. In the final Section 5 we discuss possible implications and extensions.

2 Notation and Preliminaries

We are given an (unknown) polygon P with n vertices. The length of P's boundary is denoted by D (in case of a simple polygon, D is the perimeter of P).

Every robot in the swarm has a (circular) communication range r. Within this range, each robot can perceive other robots and communicate with them. For the ease of description we assume that r is equal to 1 (and scale the polygon accordingly).

Given an unknown polygon P, the *Online Minimum Relay Triangulation Problem* (OMRTP) asks for a triangulation of P that covers P. The triangulation must not contain edges crossing the boundary of P, reflecting the impossibility to communicate through walls. The triangulation therefore contains all vertices of P, plus a number of relay points. The latter are needed because edges in the triangulation must not have a length exceeding r. The objective is to minimize the number of robots; that is, vertices of

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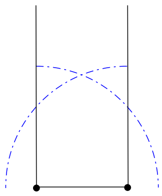

Figure 2: The polygonal corridor of width 3/4. The dash-dotted lines indicate parts of the transmission ranges of the relays located at the vertices.

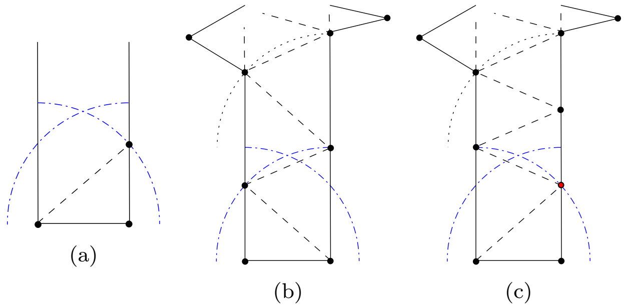

Figure 3: In case the algorithm places the next relay on the boundary (w.l.o.g. on the right boundary) (a), the optimum needs 8 relays (b), the algorithm 9 (c). The dashed lines indicate the edges of the triangulation.

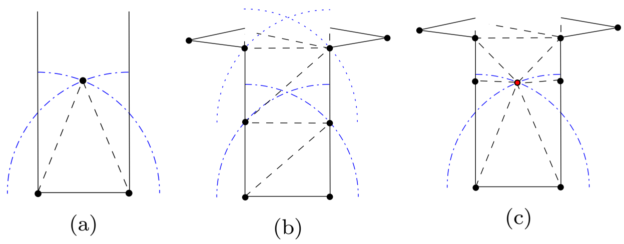

Figure 4: In case the algorithm places the next relay in the center (a), the optimum needs 8 relays (b), the algorithm 9 (c). The dashed lines indicate the edges of the triangulation.

the triangulation. This is equivalent to minimizing the number of relays. Let R_{OPT} denote the number of relays used by the optimum.

The robots start from a given point located at the boundary of the unknown environment. Each robot is allowed to move through the area, and will then decide a new location for a vertex of the triangulation. There it stops moving and becomes part of the static triangulation. This is motivated in the application: It is desirable to partially fix the triangulation as it is constructed, to begin location services in this area even if the polygon is not fully explored yet. This is a crucial property if we assume a huge area that is explored over long times, determined by the rate of new robots being added to the system.

3 Lower Bound

For the lower bound we use a polygonal corridor of width 3/4, see Figure 2. For a complete triangulation, relays must be placed at the vertices, i.e., the position of the first two relays is fixed.

In case the algorithm places the next relay on the boundary (w.l.o.g. we assume that it places the relay on the right boundary, otherwise a mirrored construction is used), the polygonal corridor will be constructed as depicted in Figure 3(b). Hence, the optimum needs 8 relays, the algorithm uses 9.

If, on the other hand, the algorithm locates the next relay in the center, see Figure 4(a), the polygonal corridor will be constructed as depicted in Figure 4(b). Consequently, the optimum needs again 8 relays for the triangulation, while the algorithm uses 9.

An arbitrary number of these polygonal pieces can be joined using small triangular structures as depicted in Figures 3(b), 3(c), 4(b) and 4(c), as the vertices require relays. Thus, we have:

Theorem 1 *No deterministic algorithm for the online minimum relay triangulation problem can be better than $\frac{9}{8}$-competitive.*

4 Online Triangulation

In the following, we describe our algorithm for the online minimum relay triangulation problem. We split the construction in two parts: (i) a triangulation along P's boundary and (ii) a triangulation of the interior.

For the boundary (i.e., the polygon's outer boundary and the boundary of holes in the environment) we place relays within distance 1 along the boundary and on vertices. Furthermore, we add a "second layer" along the boundary by placing relays within a distance of (at most) $\frac{\sqrt{3}}{2}$ to the boundary, and within distance of (at most) 1 to the relays located on the boundary, see Figure 5. Assuring this triangulated layer of width $\frac{\sqrt{3}}{2}$ also at vertices, we need to have a closer look at reflex vertices. The critical case for placing many relays arises from a reflex vertex with interior angle close to 360°, see Figure 7. Thus, the maximum number of additional relays located at a reflex vertex is 3, while we do not need any additional relays at non-reflex vertices. So, we add at most $3n$ relays. Consequently, the triangulation along P's boundary does not use more than $2D + 3n$ relays.

We still need to take care of (ii): a triangulation of the interior. Parts of the polygon of width less than $\sqrt{3}$ are already covered, so we do not need to take these into account, see Figure 8. For the remaining polygon we use a triangular point grid with side length 1, see Figure 6. In case points of this grid would lie inside the triangulated $\frac{\sqrt{3}}{2}$-layer along P's boundary, they get dragged outside of this layer, as depicted in Figure 9, in order to assure that the re-

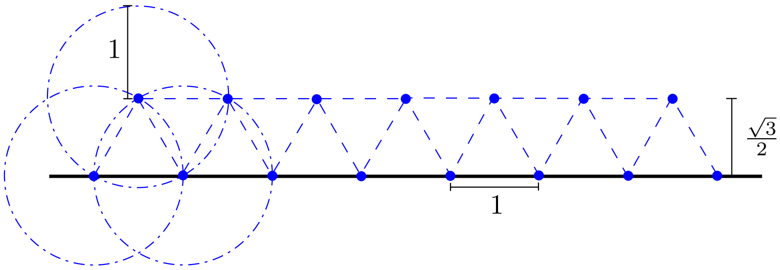

Figure 5: Example for the triangulation along the boundary, with cost at most $2D$. The bold line represents the boundary of P, the dashed lines indicate the edges of the triangulation, the dashed-dotted lines indicate again the communication range for some relays.

sulting two parts can be glued into one triangulation (i.e., in order to assure a distance of at most $r = 1$). In case points of this triangular point grid would coincide with the points from the boundary construction, they will simply not be placed, so that no degenerate triangles occur. Let k be the number of relays used for this construction.

Altogether, we have:

$$R_{\text{ALG}} \leq k + 2D + 3n \tag{1}$$

On the other hand, we can establish lower bounds on the number of relays for an optimal solution for OMRTP (i.e., lower bounds for R_{OPT}).

First, an easy observation yields the lower bound in (2): For a complete triangulation of P we need to place a relay on every vertex of the polygon.

$$R_{\text{OPT}} \geq n \tag{2}$$

Moreover, the triangulation needs to establish edges along all edges of the polygon P. As the maximum distance of relays is $r = 1$, we have:

$$R_{\text{OPT}} \geq D \tag{3}$$

As described earlier, k is the number of relays used when overlaying P with a triangular point grid with side length 1 such that all points are located inside of P (i.e., in the interior, on edges or on vertices), see

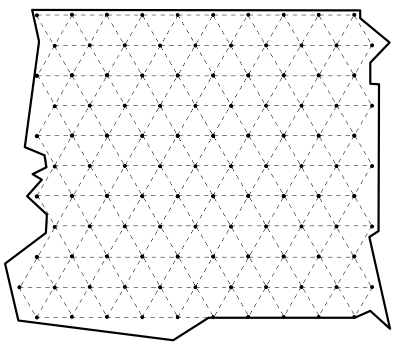

Figure 6: Example for a cover of the interior of P, using k relays.

Figure 6. Obviously, k is not uniquely defined, but for any such overlay the optimum cannot use less than k relays to triangulate P, i.e., we get a lower bound of

$$R_{\text{OPT}} \geq k \tag{4}$$

Combining Equations 1–4 yields:

Theorem 2 *There is a 6-competitive strategy for the online minimum relay triangulation problem in polygons (even with holes).*

Note that we can find the places for relays in an online fashion: From the given starting point, relays move along the boundary, assuring the placement of the relays for the triangulation along P's boundary. Then, again starting from the given start point, an overlay with a triangular point grid is constructed. local adjustments of the type in Figure 9 assure the placements of the relays in accordance with the strategy. When a hole is encountered during the construction of the triangular grid for the interior the boundary, a $\frac{\sqrt{3}}{2}$-layer is constructed around the hole.

Simple Polygons. We can achieve a factor better than 6 for simple polygons by a more careful analysis of the relays that we place at vertices. Let n_{240} be the number of reflex vertices whose interior

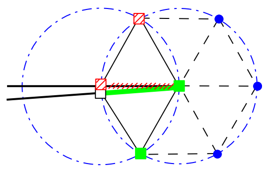

Figure 7: Example for the triangulation at a 360° reflex vertex. The black semi-bold lines represent the boundary of P (for clarity the two boundary lines are not drawn parallel but one is slightly offset from its actual position). The circular points are relays charged to the reflex vertex. Squares get charged to the perimeter—e.g., the shaded squares to the shaded piece of boundary and the filled squares to the bold piece of boundary. At most 3 additional relays are used.

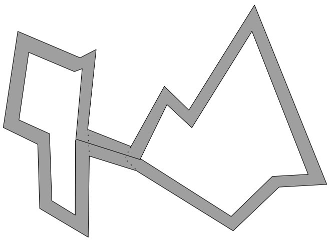

Figure 8: A polygon P. The layer of width $\frac{\sqrt{3}}{2}$ is depicted in gray.

angle, α_i, is greater than 240° and let n_{nr} be the number of non-reflex vertices. For the sum of the interior angles in a simple polygon, $\Sigma\alpha_i$, we have: $\Sigma\alpha_i = (n-2)\cdot 180° < n \cdot 180°$. This equation yields a bound on n_{240}:

$$\begin{aligned} \Sigma\alpha_i &\geq n_{240}\cdot 240° + n_{\mathrm{nr}}\cdot 0° \\ &\quad +(n-n_{\mathrm{nr}}-n_{240})\cdot 180° \\ \Leftrightarrow \quad (n-2)\cdot 180° &\geq (n-n_{\mathrm{nr}})\cdot 180° + n_{240}\cdot 60° \\ \Leftrightarrow \quad (n_{\mathrm{nr}}-2)\cdot 3 &\geq n_{240} \\ \Rightarrow \quad n_{240} &\leq 3\cdot n_{\mathrm{nr}} \end{aligned}$$

For each of the n_{240} reflex vertices we need at most 3 additional relays, for the n_{nr} non-reflex vertices no additional relays are placed. For each of the remaining $(n-n_{\mathrm{nr}}-n_{240})$ reflex vertices (with an interior angle $\leq 240°$) we place one additional relay. Hence, for the total number of relays added at vertices, R_{V}, we have:

$$\begin{aligned} R_{\mathrm{V}} &\leq (n-n_{\mathrm{nr}}-n_{240})\cdot 1 + n_{240}\cdot 3 + n_{\mathrm{nr}}\cdot 0 \\ &\leq (n-n_{\mathrm{nr}})\cdot 1 + 6\cdot n_{\mathrm{nr}} = n + 5\cdot n_{\mathrm{nr}} \end{aligned}$$

We distinguish two cases as follows.

1. $n_{\mathrm{nr}} \leq \frac{n}{4} \Rightarrow R_{\mathrm{V}} \leq n + \frac{5}{4}n = \frac{9}{4}n$
2. $n_{\mathrm{nr}} > \frac{n}{4}$. In this case, we need at most 3 additional relays for at most $\frac{3}{4}$ vertices: $n - n_{\mathrm{nr}} \leq \frac{3}{4}n \Rightarrow R_{\mathrm{V}} \leq 3\cdot\frac{3}{4}n = \frac{9}{4}n$

Altogether, we have:

$$R_{\mathrm{ALG}} \leq k + 2D + \frac{9}{4}n \leq \frac{21}{4}R_{\mathrm{OPT}} \tag{5}$$

Theorem 3 *A simple polygon allows a $\frac{21}{4}$-competitive strategy for the online minimum relay triangulation problem.*

5 Conclusion

We introduced the online minimum relay triangulation problem. We gave a lower bound of $\frac{9}{8}$ for the competitive ratio for any online algorithm (even in simple polygons). For polygons we presented a 6-competitive algorithm, and showed that it is $\frac{21}{4}$-competitive for simple polygons. Considering the gap between the lower bound and the given competitive ratio an open question is whether we are able to improve the ratio.

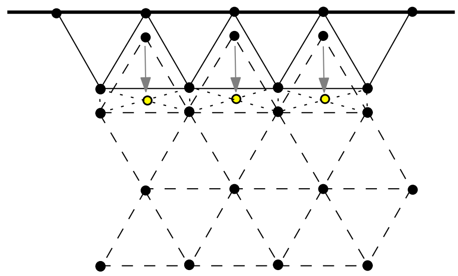

Figure 9: An example for the combination of the triangulation along P's boundary and the triangulation of the interior into one triangulation.

Another closely related problem arises from considering a limited number of robots; the *Maximum Coverage Triangulation Problem* (OMCTP) asks for a triangulation using ℓ robots, such that the area within an unknown polygon P that is covered is maximized.

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